Multilinear Regression

Prof Wells

STA 295: Stat Learning

February 13th, 2024

Outline

In today's class, we will. . .

- Generalize the simple regression model to include more than 1 predictor
- Quantify model accuracy for linear regression models (both simple and multiple)
- Implement multiple regression in R

Section 1

Multiple Regression

Many Simple Linear Regression Models?

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- And even if none of the predictors have strong association with the response, we are likely to observe a significant predictor just due to chance.

Could we get better predictive power by including all explanatory variables in the *same* model?

Multiple Regression Model

In a **simple linear regression model** (SLR), we express the response variable Y as a linear function f of one predictor variable X:

$$Y = f(X) + \epsilon$$

and estimate f using

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- In the MLR model, we allow predictors to either be quantitative or binary categorical (i.e taking values 0 or 1 corresponding to failure or success)

Finding Parameters

To create an SLR model, we found the equation of a line that minimizes RSS, where

$$\mathrm{RSS} = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} \Big(y_i - (\beta_0 + \beta_1 x_1) \Big),$$

which has the solution

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To fit an MLR model. . .

we do the exact same thing!

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- $\mathbf{y}$ is the n-vector of observed responses
- $\mathbf{X}$ is the matrix (or dataframe) consisting of n rows of observations on p predictors (along with a column of 1's to create an *intercept*).

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- In the SLR case with 1 predictor, the estimates $\hat{\beta}_0, \hat{\beta}_1$ describe a line in 2D space:

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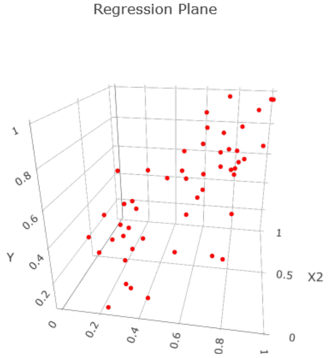
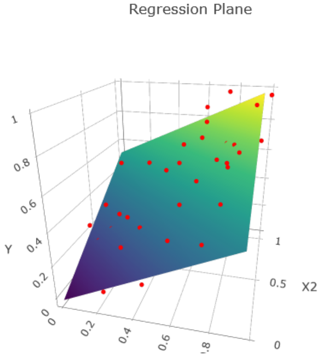
- But if we have 2 predictors, the estimates $\hat{\beta}_0, \hat{\beta}_1, \hat{\beta}_2$ describe a plane in 3D space

$$y = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2$$

The Plane of Best Fit





An interactive graphic available under topics list for Tuesday 2-13 on course website

Example: Credit Card Balance

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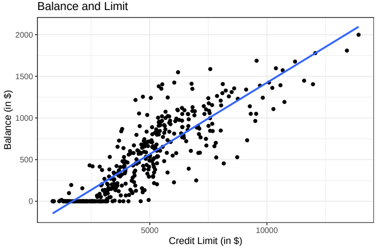

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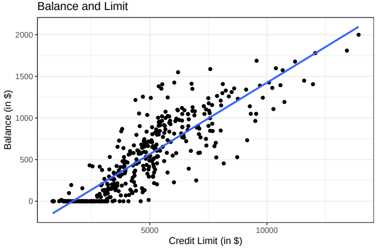

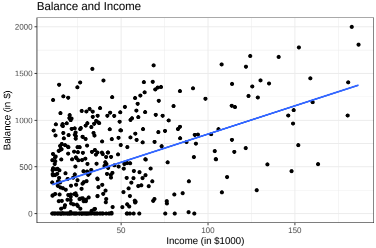

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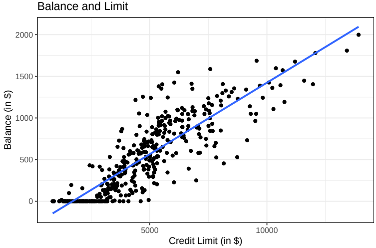

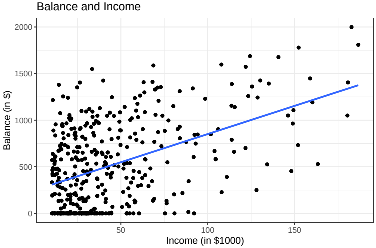

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Both variables have some explanatory power for Balance

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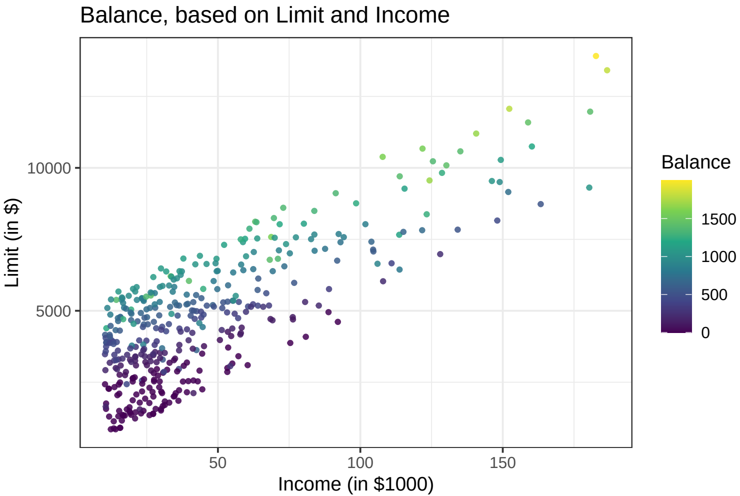

The Multilinear Visualization II

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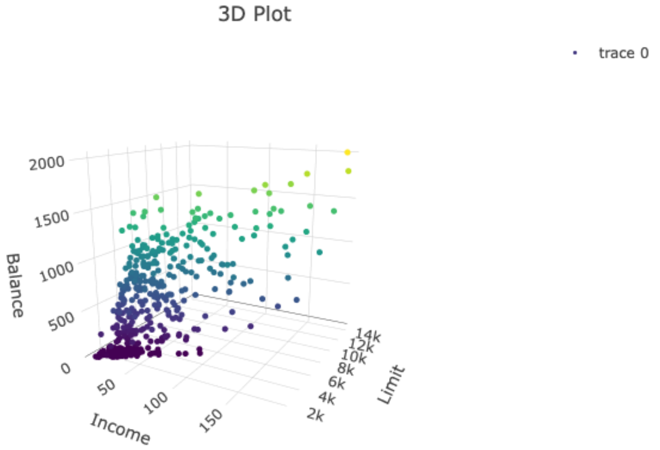

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- While for **fixed** value of Limit, increasing Income by \$1000 decreases Balance by an average of \$7.66.

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- How is this possible?

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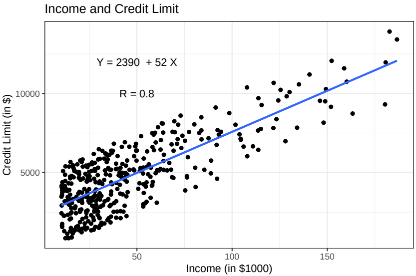

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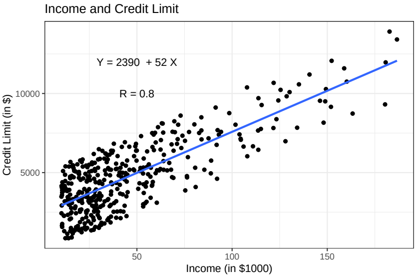

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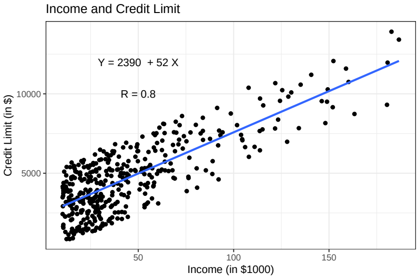

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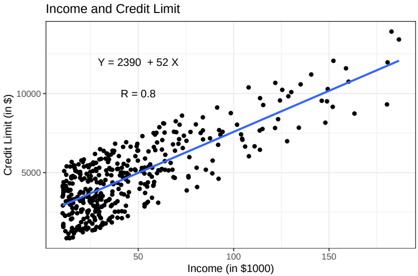

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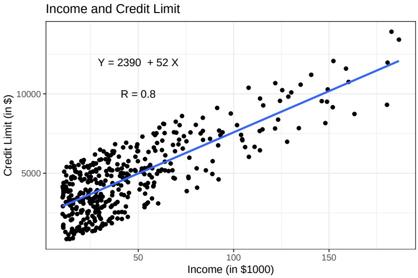

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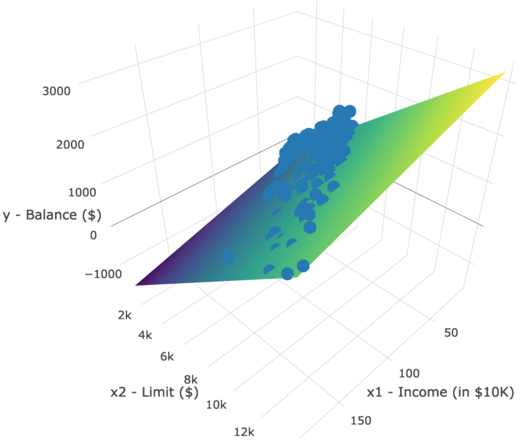

Balance vs. Income Revisited

We can lump Credit Limits into 4 brackets (low, med-low, med-high, high) to create a categorical variable and analyze the SLR for Balance and Income for each level of Credit Limit

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Section 2

Assessing Model Accuracy

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- Recall **Residual Standard Error** (RSE) measures the average size of deviations of the response from the linear regression line. It is given by

$$\mathrm{RSE} = \sqrt{\frac{1}{n-1-p}\mathrm{RSS}} = \sqrt{\frac{1}{n-1-p}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

How Strong is a Linear Model?

- In an linear model model, $Y = f(X) + \epsilon$. Even if we could perfectly predict f using $\hat{f}$, our model would still have non-zero MSE.
- Recall **Residual Standard Error** (RSE) measures the average size of deviations of the response from the linear regression line. It is given by

$$\mathrm{RSE} = \sqrt{\frac{1}{n-1-p}\mathrm{RSS}} = \sqrt{\frac{1}{n-1-p}\sum_{i=1}^{n}(y_i - \hat{y}_i)^2}$$

- It has the property that

$$\mathrm{Avg}(\mathrm{RSE}^2) = \mathrm{Var}(\epsilon)$$

- Which means that $\mathrm{Abvg}(\mathrm{RSE}) \approx \mathrm{sd}(\epsilon)$

Five Flavors of Error

Which of the following are most likely to **decrease** as more and more predictors are added to a linear model (select all that apply)?

a. test MSE
b. training MSE
c. RSS
d. RSE
e. $\text{Var}(\epsilon)$

The R^2 statistic

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- The value of R^2 is always between 0 and 1, and represents the percentage of variability in values of the response just due to variability in the predictors.

Values of R^2

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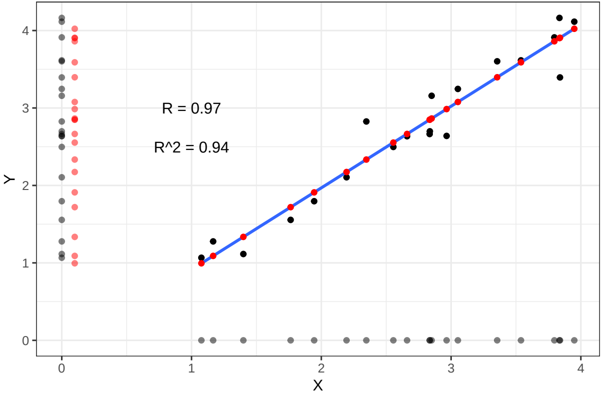

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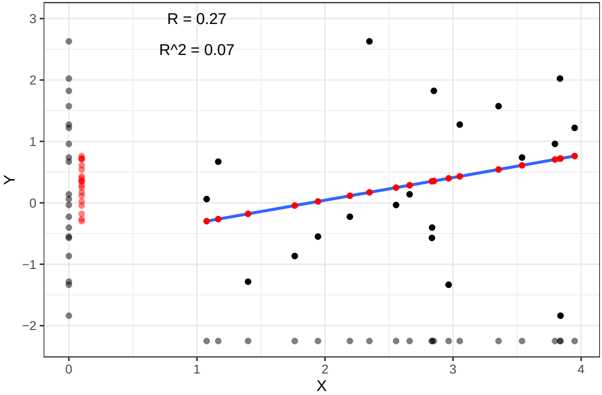

Formulas for R^2 in terms of correlation

For SLR,

$$R^2 = [\mathrm{Cor}(X, Y)]^2 = \left[\frac{\mathrm{Cov}(X, Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} \right]^2 = \left[\frac{\sum_{i=1}^n (x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^n (x_i - \bar{x})^2} \sqrt{\sum_{i=1}^n (y_i - \bar{y})^2}} \right]^2$$

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We will usually use software to compute R^2.

Model Accuracy in R

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summary(mod_credit)
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##
## Call:
## lm(formula = Balance ~ Income + Limit, data = Credit)
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## Residuals:
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## Income        -7.66332    0.38507  -19.90   <2e-16 ***
## Limit          0.26432    0.00588   44.95   <2e-16 ***
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
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## Residual standard error: 165.5 on 397 degrees of freedom
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We can use `summary(mod)$r.sq` or `summary(mod)$sigma` to access R^2 and RSE directly.

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  - The formula for adjusted R^2 depends on RSS, which is computed on **training** data.

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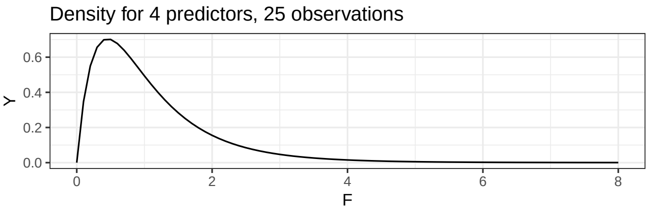

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Moreover, it is unlikely that F is drastically larger than 1.

Poll: TSS and RSS (optional)

Suppose we have a linear model with 25 observations and 4 predictors. Which of the following provides the best evidence of a relationship between the response and at least 1 of the predictors?

a. $\text{TSS} = 64$, $\text{RSS} = 4$
b. $\text{TSS} = 4$, $\text{RSS} = 16$
c. $\text{TSS} = 48$, $\text{RSS} = 8$
d. $\text{TSS} = 4$, $\text{RSS} = 4$

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    - Yes. But we'll cover detailed model selection in Chapter 6.